CSE 203A: Advanced Algorithms

Winter 2015 — Home work 1

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Exercise 1.24: Finding minimum r-way cut set

A *r-way cut* $(A_1, A_2, \dots A_r)$ of an undirected, unweighted graph $G = (V, E)$, is a partition of its nodes V into r non-empty and disjoint sets $A_1, A_2 \dots A_r \subset V$, such that $\bigcup_i A_i = V$.

A *r - way cut-set* is the set of edges crossing the r-way cut.

The *minimum r-way cut set* is the *r-way cut set* with the minimum cardinality.

Algorithm

We will use the randomised min cut algorithm, also known as the Karger's Algorithm, to find a *minimum r-way cut.*

Algorithm We perform edge ccontraction on the graph till there are only r vertices left. So, there are a total of $|V| - r$ iterations and in each iteration the algorithm chooses an edge from the existing edges uniformly at random and contracts it. The resulting graph after $|V| - r$ iterations, is a *r-way cut* and the probability of this cut being a *min r-way cut* is analysed below.

Analysis To analyse the probability that this algorithm outputs a *min r-way cut set* we need to find the relationship between the number of edges of the graph and the min cut set. Hereon for convenience, we will use *min-cut* in general to refer to *min r-way cut* unless otherwise specified.

Let k be the total number of edges that form the *min r- way cut* of the graph and n be the number of vertices, $n = |V|$.

Let $d_1, d_2, \dots, d_n$ be the degree of the vertices such that $d_1 \leq d_2 \dots \leq d_n$. If we remove the edges incident to the vertices with degrees $d_1, d_2, \dots, d_{r-1}$ we will get a *r-way cut.* And,

$$d_1 + d_2 + \dots + d_{r-1} \geq k$$

And so the average degree of these $r - 1$ vertices,

$$\frac{d_1 + d_2 + \dots + d_{r-1}}{r - 1} \geq \frac{k}{(r - 1)}$$

Since, the rest of the vertices, with degrees $d_r, \dots, d_n$ have degree greater than or equal to these $r-1$ vertices, we can conclude that the average degree of all the n vertices is greater than the average degree of these $r-1$ vertices

$$\frac{\sum_{i=1}^{n} d_i}{n} \geq \frac{k}{(r - 1)}$$

$$\sum_{i=1}^{n} d_i \geq \frac{nk}{(r-1)}$$

And hence, for the edges of the graph

$$|E| = \frac{\sum_{i=1}^{n} d_i}{2} \geq \frac{nk}{2*(r-1)}$$

$$|E| \geq \frac{nk}{2*(r-1)}$$

Now, for the algorithm to output a r-way min cut it should not contract any of the k edges in its $n-r$ iterations.

Let E_i be the event that the edge selected in the ith iteration is not in the min cut and F_i be the event of not selecting any of the min-cut edges in the first i iterations. So, $F_i = \cap_{j \leq i} E_i$.

Hence, for the first iteration, $i = 1$:

$$\Pr\left(\text{An edge of the min -cut is choosen}\right) \leq \frac{k}{\frac{nk}{2*(r-1)}}$$

$$\Pr\left(\text{An edge of the min -cut is choosen}\right) \leq \frac{2*(r-1)}{n}$$

So,

$$\Pr(E_1) = \Pr(F_1) \geq 1 - \frac{2*(r-1)}{n}$$

For the second iteration, $i = 2$, given that none of the min-cut edges are selected in the first iteration, we have

$$\Pr\left(E_2|F_1\right) \geq 1 - \frac{2*(r-1)}{n-1}$$

$$\Pr\left(F_2\right) = \Pr\left(E_2 \cap E_1\right) = \Pr\left(E_2|E_1\right).\Pr(E_1) = \Pr\left(E_2|F_1\right).\Pr(F_1)$$

Similarly,

$$\Pr\left(E_i|F_{i-1}\right) \geq 1 - \left(\frac{2*(r-1)}{n-i+1}\right)$$

Now, for the algorithm to output the min cut, it should not select any of the cut-set edges in each of the $n-r$ iterations.

Hence,

$$\Pr\left(\text{Algorithm outputs min -cut}\right) = \Pr(F_{n-r})$$

$$\begin{aligned}
\Pr(F_{n-r}) &= \Pr(E_{n-r} \cap F_{n-r-1}) = \Pr(E_{n-r}|F_{n-r-1}).\Pr(F_{n-r-1}) \\
&= \Pr(E_{n-r}|F_{n-r-1}).\Pr(E_{n-r-1}|F_{n-r-2})\ldots\Pr(E_2|F_1).\Pr(F_1) \\
&\geq \prod_{i=1}^{n-r}\left(1 - (\frac{2*(r-1)}{n-i+1})\right)
\end{aligned}$$