Problem 2.8

(a) Stop having children until they have a girl or $k(k \ge 1)$ children. Each child is a boy or girl is independently with probability $\frac{1}{2}$ and there are no multiple births. Derive the expected number of female children they have. Derive the expected number of male children they have.

Solution:

Suppose the event of having a girl is X_g, the event of having boys is X_b.

$$E(X_g) = 1\cdot(\frac{1}{2}) + 1\cdot(\frac{1}{2})^2 + \ldots + 1\cdot(\frac{1}{2})^{k-1} + 1\cdot(\frac{1}{2})^k + 0\cdot\left(\frac{1}{2}\right)^k$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

So (k - 1)th child is a girl k - th child is a girl k - th child is a boy

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k$$

$$= \sum_{i=1}^{k} \left(\frac{1}{2}\right)^i = \sum_{i=1}^{k} (2)^{-i}$$

According to the sum of geometric sequence $S_n = \frac{a_1 - a_n q}{1-q}$

So $$E(X_g) = \frac{\frac{1}{2} - (\frac{1}{2})^k \cdot \left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^k}{\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^k$$

Then we derive the expectation of the number of boys. For example we use a sequence s_1 to represent the sequence of the births of their children, suppose $s_1 = bbbbg$, which means the 1st child is a boy, the 2nd child is a boy, ... And the last child is a girl, it is obvious to see that if we have a sequence $s_1' = bbbbb$, $\text{Prob}(s_1) = \text{Prob}(s_1') = \left(\frac{1}{2}\right)^5$, then after s_1', suppose $s_2 = bbbbbb$, $s_2' = bbbbbg$, which indicates that $\text{Prob}[b \mid s_1] = \text{Prob}[g \mid s_1]$, since the probability of having either a boy or a girl is equal to $\frac{1}{2}$. Applying the same principle, we can deduce the more general expression of $E(X_b)$ and $E(X_g)$. According to conditional probability, we have:

$$E(X_b) = \sum_s E[b \mid s] = \sum_s \text{Prob}[b \mid s]$$
$$E(X_g) = \sum_s E[g \mid s] = \sum_s \text{Prob}[g \mid s]$$

So according to mathematical induction, we have $E(X_b) = E(X_g) = 1 - \left(\frac{1}{2}\right)^k$

(b) Stop having children only when they have a girl. Derive the expected number of boys that they have.

Solution:

Suppose Bob and Alice keep having children until they have a girl is event I.

This event is the union of many sub-events, namely,

1. The first child they have is a girl, so the number of boys they have is 0, with probability $\frac{1}{2}$.
2. The second child they have is a girl, so the number of boys they have is 1, with probability $\frac{1}{2} \times \frac{1}{2}$.
3. .....
4. ....

....

n-1. The n-th child they have is a girl, so the number of boys they have is $n-1$, with probability $(\frac{1}{2})^{n-1} \cdot (\frac{1}{2}) = (\frac{1}{2})^n$

Since each sub-event is mutual exclusive, the expectation of event I can be written as:

$$E(I) = 0 \cdot (\frac{1}{2}) + 1 \cdot (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2})^3 + \ldots + (n-1) \cdot (\frac{1}{2})^n$$

Observe this equation, we see that it is similar to the equation of $E(X_b)$ in problem (a), since $\lim_{n \to \infty} (1 - 2^{-n}) = 1$, so in this question when $n \to \infty$, $E(I) = 1$.