# Parity

## 1 Problems

**Problem 1: Last Ball**

1. You have $20$ black balls and $16$ white balls in a bag. You repeat the following operation until a single ball is left in the bag. You remove two balls at a time. If they are of the same color, you add a black ball to the bag; if they are of different colors, you add a white ball to the bag. Can you predict the color of the last ball left in the bag?
2. Answer the same question if there are $20$ black balls and $15$ white balls to start with.

**Problem 2: Remaining Number**

The first $50$ natural numbers — $1, 2, \ldots, 50$ — are written on a board. You have to apply the following operation $49$ times: select two of the numbers on the board, $a$ and $b$, write the absolute value of their difference $|a - b|$ on the board, and then erase both $a$ and $b$. Determine all possible values of the remaining number that can be obtained in this manner.

**Problem 3: Pluses and Minuses**

The $n$ consecutive integers from $1$ to $n$ are written in a row. Design an algorithm that puts signs '$+$' and '$-$' in front of them so that the expression obtained is equal to $0$ or, if the task is impossible to do, returns the message no solution. Your algorithm should be much more efficient than an examination of all possible ways to place the signs.

**Problem 4: Chips on Sectos**

A circle is divided into $n > 1$ sectors, and one chip is placed on each of them. A move is made by moving two chips to their neighboring sectors (in the same or opposite directions). For which values of $n$ is there an algorithm to collect all the chips on the same sector?

**Problem 5: Upside-Down Glasses**

There are $n$ glasses on the table, all standing upside down. In one move, you are allowed to turn over exactly $n - 1$ of them. Determine all values of $n$ for which all the glasses can be turned up, and outline an algorithm that does this in the minimum number of moves.

**Problem 6: Reversal of Sort**

There are $n$ index cards in a row, with $n$ distinct integers written on them (one number per card) so that the numbers are sorted in decreasing order. You are allowed to exchange any pair of cards that have exactly one card between them. For which values of $n$ is it possible to make the cards sorted in increasing order with a sequence of such operations? When it is possible, indicate an algorithm with the minimum number of exchanges.

**Problem 7: Fox and Hare**

Consider a chase game that we call The Fox and the Hare. The game is played on a one-dimensional board with $30$ cells numbered left to right from $1$ to $30$. A chip representing the fox starts at cell $1$, and a chip representing the hare starts at some cell $s > 1$. They move alternately, with the fox moving first. On each move, the fox can move left or right to a neighboring cell; the hare jumps left or right over two cells landing on the third. The hare cannot land on a cell occupied by the fox; if he does not have another move, he loses the game. And, of course, neither of them can move outside the board. The foxs goal is to catch the hare, which he can do if they occupy adjacent cells on the foxs move; the hares goal is to avoid the capture. Find all the values of $s$ for which the fox can win the game.

## 2 Challenge Problems

**Problem 8: Fifteen Puzzle**

This famous puzzle consists of fifteen square tiles numbered from $1$ to $15$ which are placed in a $4 \times 4$ box leaving one square out of the sixteen empty. The goal is to reposition the tiles from a given starting arrangement by sliding them one at a time into the configuration in which the tiles are ordered sequentially. Is it possible to solve the puzzle for the initial configuration shown in Table 1?

Table 1: Initial position of the fifteen puzzle

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 15 | 14 | |

Table 2: Final position of the fifteen puzzle

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | |